Introduction to moduli spaces

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Part I

What is a moduli space?

Moduli spaces

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Roughly, a moduli space is a (topological, geometric, algebraic) space whose points are in one to one correspondence with geometric objects of one kind.

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the moduli space M is $\mathbb{R}^4 - \{((x_0, y_0), (x_0, y_0))\} \cong \mathbb{R}^4 - \Delta \cong \mathbb{R}^4 - \mathbb{R}^2$.

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M is not only a set, but is a topological space.

Toy examples

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Define $U \subset M \times \mathbb{R}^2 \cong \mathbb{R}^4 \times \mathbb{R}^2$ by

$$U = \{(x_0, y_0, x_1, y_1, (1-t)x_0 + tx_1, (1-t)y_0 + ty_1) | t \in [0,1]\}$$

There is a natural map $\pi : U \to M$ defined by projection to first four coordinates.

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Q. Why U is a 'universal family'?

A. For every point $((x_0, y_0), (x_1, y_1)) = p \in M$,

$$\pi^{-1}(p) = \{((1-t)x_0 + tx_1, (1-t)y_0 + ty_1) | t \in [0,1]\} \subset \mathbb{R}^2,$$

the oriented line segment corresponded p!

Definition of moduli space

Definition

A (fine) moduli space of some geometric objects consists of a moduli space M, a universal family U and a map $\pi : U \to M$ such that

1. there is one to one correspondence between points of M and geometric objects we want to collect.
2. for every point $p \in M$, $\pi^{-1}(p)$ is the corresponded object.

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Define an equivalence of oriented line segments as following:
$l_1, l_2 \subset \mathbb{R}^2$ are isomorphic if $\exists$ a translation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(l_1) = l_2$.

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Then we can assume the starting point of line segment is the origin.
So in this case, the moduli space M'' of oriented line segments in $\mathbb{R}^2$ up to translation is

$$M'' = \mathbb{R}^2 - \{(0,0)\}.$$

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Lesson : Equivalent relations between parameterized objects are very important!

More examples

Fix a vector space $V = \mathbb{R}^n$.

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- every $v \in V - \{0\}$ determines a unique 1-dimensional subspace $L = \langle v \rangle \subset V$.
- $v, v' \in V - \{0\}$ determines the same subspace if $\exists c \in \mathbb{R}^*$, $v = cv'$.

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- $v, v' \in V - \{0\}$ determines the same subspace if $\exists c \in \mathbb{R}^*$, $v = cv'$.

$\Rightarrow$ the moduli space of 1-dimensional subspace of V is $(V - \{0\})/\mathbb{R}^*$ $= P(V)$, the projective space!

More examples

Moreover, there is the universal family(in this case, universal subspace) U over $P(V)$.

$$U = \{([L], v) | v \in L\} \subset P(V) \times V$$

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For $[L] \in P(V)$, $\pi^{-1}([L]) = \{([L], v) | v \in L\} \cong L$.

So $P(V)$ with $\pi : U \to P(V)$ is the fine moduli space of 1-dimensional subspaces of V.

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Exercise : $G(n-1, V) \cong P(V)$.
Sketch : Fix a positive definite inner product on V.
Define a map

$$\begin{array}{rcl} P(V) & \rightarrow & G(n-1, V) \\ L & \mapsto & L^{\perp} \end{array}$$

Check this map is bijective.

Some technical issues

If we study moduli spaces in algebraic geometry, there are two important assumptions.

- Usually, we use algebraic closed field $\mathbb{C}$ instead of $\mathbb{R}$.
- We hardly use affine space($\mathbb{C}^n$).
  Usually we use projective space $\mathbb{P}^n$.
  It is a compactification of $\mathbb{C}^n$.

Some famous moduli spaces

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C : a nonsingular curve of genus $g \geq 2$.
$M(C, r, \alpha)$: moduli space of stable vector bundles of rank r and first Chern class α.

Part II

Why we study moduli spaces?

1. Examples of higher dimensional variety

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Is it smooth? compact? connected? nonempty?

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So moduli spaces gives a plenty of examples of relatively concrete but not obvious higher dimensional algebraic objects.

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Suprisingly, the answer of last question is neither of 0, 1 nor ∞.

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$U := \{(L, v) \in G(2, V) \times V | v \in L\}$ $\cdots$ universal family.
$U^* := \{L, v) \in G(2, V) \times V | v \in L, v \neq 0\} \subset U$

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$U^* := \{L, v) \in G(2, V) \times V | v \in L, v \neq 0\} \subset U$
Exist two natural maps:

$$\begin{array}{rccc} \pi : & U^* & \to & G(2, V) \\ & (L, v) & \mapsto & L \end{array}$$

$$\begin{array}{rccc} f : & U^* & \to & V \to \mathbb{P}^3 \\ & (L, v) & \mapsto & v \mapsto \langle v \rangle \end{array}$$

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$= $ set of pairs (L, v) such that $v \in L \cap L_i$.

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$$|\bigcap_{i=1}^{4} \pi(f^{-1}(L_i))|$$

is what we want!

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$$|\bigcap_{i=1}^{4} \pi(f^{-1}(L_i))| = \int_{G(2,V)} \sigma_1^4 = 2.$$

There exist exactly 2 lines in $\mathbb{P}^3$ meet 4 general lines.

Lines on a Calabi-Yau 3-fold

Definition

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Question

How many lines in general quintic threefold?

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Lines on a Calabi-Yau 3-fold

Consider the moduli space of lines in $\mathbb{P}^4$: $Gr(2, V)$ where $V \cong \mathbb{C}^5$.
Let U be the universal vector space over $Gr(2, V)$,
and let U^* be the complement of zero section.

We have a following diagram

$$\begin{array}{ccc} U^* & \xrightarrow{f} & \mathbb{P}^4 \\ \downarrow{\scriptstyle \pi} & & \\ G(2, V). & & \end{array}$$

as before.

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Idea : Make a vector bundle W on $Gr(2, V)$ such that for $L \in Gr(2, V)$, the fiber W_L is the vector space of degree 5 homogeneous polynomials over $L \cong \mathbb{P}^1$

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Then for each line $L \subset \mathbb{P}^4$, we can restrict g to the line L and get an element g_L of W_L.

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Then for each line $L \subset \mathbb{P}^4$, we can restrict g to the line L and get an element g_L of W_L.
Moreover, $g_L = 0$ iff L is in the zero set $Z(g) = X$.

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Let our quintic 3-fold X is defined by a degree 5 homogeneous polynomial g of degree 5.
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So we have a section s of W, and the number of lines in $X = |Z(s)|$.

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Algebraic construction of W :

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$$\int_{G(2,V)} c_6(\pi_* f^* \mathcal{O}(5))$$

Lines on a Calabi-Yau 3-fold

Algebraic construction of W :

$$\begin{array}{ccc} \mathbb{P}(U) & \xrightarrow{f} & \mathbb{P}^4 \\ \downarrow{\scriptstyle \pi} & & \\ G(2, V). & & \end{array}$$

$\mathcal{O}(5)$: a line bundle such that one of section is g.
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By using Riemann-Roch theorem, we can compute this number : $2,875$.

Lines on a Calabi-Yau 3-fold

Clemens' conjecture

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It is proved for $n \leq 7$.

degree	number of curves
1	2,875
2	609,250
3	317,206,375
4	242,467,530,000
5	22,930,588,887,625
6	248,249,742,118,022,000
7	295,091,050,570,845,659,250

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For genus 2, the following equation is a universal equation.

$$y^2 = x^6 + a_5 x^5 + \cdots + a_1 x + a_0.$$

Existence of almost universal equations

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For any genus g, can we find a universal(or an almost universal) equation(or equations) with almost free variable for genus g curves?

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This is an immediate corollary of following theorem:

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This is an immediate corollary of following theorem:

Theorem (Harris-Mumford, Farkas)

For $g \geq 22$, the moduli space M_g is of general type.

4. Essential elements in modern mathematics and physics

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The Gromov-Witten invariant and quantum cohomology are parts of superstring theory.

Part III

Moduli spaces and birational geometry

Compactification of moduli spaces

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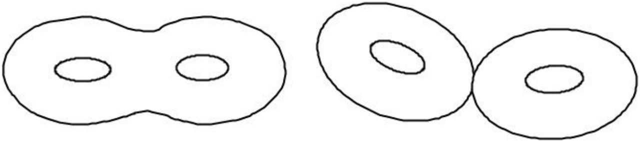
Example :
1) $\overline{M}_g$: moduli space of curves of (arithmetic) genus g with nodal singularities which has finite automorphism group.



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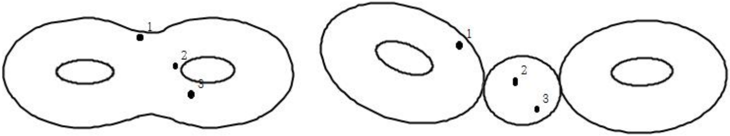

3) Compactify $M_g(\mathbb{P}^r, d)$ $\cdots$ regard $C \subset \mathbb{P}^r$ as an injective morphism $f : C \hookrightarrow \mathbb{P}^r$ from smooth curve C of genus g.
$\overline{M}_g(\mathbb{P}^r, d)$: moduli space of maps from a nodal curve of (arithmetic) genus g to $\mathbb{P}^r$ (called moduli space of stable maps), such that
i) degree of map is d,
ii) automorphism group of map is finite.

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$\cdots$ called moduli space of weighted stable curves.
gives different compactification of $M_{g,n}$.

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We say two algebraic varieties M_1, M_2 are birational if they have open dense subsets U_1, U_2 respectively, such that $U_1 \cong U_2$.

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1) Is there any morphism $\overline{M}_{g,\mathcal{A}} \to \overline{M}_{g,\mathcal{B}}$ for two different weights $\mathcal{A}$, $\mathcal{B}$?
2) Is there any other construction of $\overline{M}_{g,\mathcal{A}}$?

Digression to minimal model program

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So we want to classify minimal surfaces.

There are many works about classification of minimal surfaces(Enriques, Kodaira, ...).

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3. There is another kind of surgery, called flip.
4. The existence of minimal model is still unknown except some special cases(for example, 3-folds, log Fano, ...).
5. If a minimal model exists, there is a unique (log) canonical model.

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Theorem (Alexeev-Swinarski, Kiem-M)

Let α be a rational number satisfying $\frac{2}{n-1} < \alpha \leq 1$. Then the log canonical model $\overline{M}_{0,n}(\alpha)$ for $(\overline{M}_{0,n}, \alpha D)$ satisfies the following:

1. *If $\frac{2}{\lfloor \frac{n}{2} \rfloor + 1} < \alpha \leq 1$, then $\overline{M}_{0,n}(\alpha) \cong \overline{M}_{0,\mathcal{A}_\alpha}$ where $\mathcal{A}_\alpha = (\epsilon_\alpha, \epsilon_\alpha, \cdots, \epsilon_\alpha)$.*
2. *If $\frac{2}{n-1} < \alpha \leq \frac{2}{\lfloor \frac{n}{2} \rfloor + 1}$, then $\overline{M}_{0,n}(\alpha) \cong (\mathbb{P}^1)^n//SL(2)$. where // means some kind of algebraic group quotient(GIT quotient).*

Thank you!